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ON EMBEDDING OF INTEGRABLE EQUATIONS IN (1 + 1) AND (2 + 1) DIMENSIONS INTO THE GENERALIZED SELF-DUAL YANG — MILLS EQUATIONS

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The generalization of the self-dual Yang — Mills (SDYM) equations on the spaces of arbitrary even dimension is considered. It is shown that all integrable equations in (1 + 1) dimensions and many integrable equations in (2 + 1) dimensions may be obtained by the reduction of the generalized SDYM equations.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

О вложении интегрируемых уравнений в (1 + 1) и (2 + 1) измерениях в обобщенные уравнения автодуальности модели Янга — Миллса

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Рассмотрено обобщение уравнений автодуальности модели Янга — Миллса на пространства произвольной четной размерности. Показано, что все интегрируемые уравнения в (1 + 1) измерениях и многие интегрируемые уравнения в (2 + 1) измерениях могут быть получены редукцией обобщенных уравнений автодуальности модели Янга — Миллса.

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1. It is known, that many integrable equations in (1+1) dimensions may be embedded into the SDYM equations in $d = 4$ dimensions (see, e.g., [1—7]). This is connected with the fact that SDYM equations may be written as a compatibility condition of two linear equations with the spectral parameter $\lambda \in C$ [8]. Imposing symmetries and algebraic constraints to the fields involved permits one to reduce SDYM equations to the Korteweg — de Vries (KdV) equations, generalized nonlinear Schrödinger (NLS) equations, Boussinesq and many others having a zero curvature representation

$$\partial_t U(\lambda) - \partial_x V(\lambda) + [U(\lambda), V(\lambda)] = 0.$$

Here matrices U and V are polynomials of λ of degree not higher than a second, or functions of $\frac{1}{\lambda \pm 1}$ (chiral models, for example). Clearly, the derivative NLS equations, the Landau — Lifshitz equations and many others, having another type of dependence on spectral parameter, can't be embedded into the $d = 4$ SDYM equations. The hierarchies generated by the equations considered in [2,6] (KdV, NLS, AKNS, DNLS and other hierarchies) also are not embedded into them. That is why the SDYM equations in $d = 4$ can't play the role of the universal integrable system.

2. To solve these problems, it was suggested to consider the generalized SDYM equations for $d > 4$. Such equations were considered by Salamon [9], Ward [10], Galperin, Ivanov, Ogievetsky and Sokatchev [11] and by many others. The main progress was made by considering the self-duality equations in $d = 4n$, in which the hierarchies of KdV, NLS, DNLS, AKNS and of other equations may be embedded [2,6].

It is interesting to note that the geometric definition of self-duality in terms of linear systems and complex structure on R^{4n} (see [9—11]) are equivalent to the algebraic definition of self-duality (see, e.g., [12—15]). It was pointed by Strachan [6], how one may embed a number of hierarchies in (2 + 1) dimensions into these equations. But in all these approaches one obtains only the rational dependence on the spectral parameter λ, and it is not clear how to include into consideration the models with the spectral parameter λ that belongs to the surfaces of genus $g \geq 1$. That is why such important equation as Landau — Lifshitz equation [16] is out of consideration.

We shall show a way to overcome this difficulty.

3. Method of solving of SDYM equations in $d = 4$ is connected with the ideas of the twistor theory [17]. The SDYM equations in $d = 4k$ are connected with the twistor theory for $4k$-dimensional hyper-Kähler manifolds [9—11,18]. Further generalization of the twistor theory (and of the self-duality equations) was considered in [19].

So for any Riemannian even-dimensional manifold M^{2n} we may consider a bundle $j(M^{2n})$ of the Riemannian almost complex structure with fibers $F = SO(2n)/U(n)$. The idea of the papers [19] is that we may choose as a twistor manifold a submanifold Z in $j(M^{2n})$ with fibres $B \subset SO(2n)/U(n)$. In these papers the case of $B = G/H$ and, in particular, of $B = CP^1 = Sp(1)/U(1)$ is considered as an example. But as B we may also choose the Riemannian surfaces of genus $g \geq 1$, and, in particular, the elliptic curves. They are embedded into the fibres $SO(2n)/U(n)$ with $n \geq 3$ over the $2n$-dimensional Riemannian manifold. We may use this fact.

Let us consider the flat case of R^{2n} and $j(R^{2n}) \cong R^{2n} \times F$. We have a bundle $j(R^{2n}) \to F$, where $F = SO(2n)/U(n)$. This is a canonical universal complex bundle, geometry of which is well known (see, e.g., [20]). The fibre C_J^n over a point $J \in F$ is identified with the complex vector space (R^{2n}, J) of dimension n. Let us consider for simplicity one coordinate patch on F. Coordinates on it we may identify with the antisymmetric $n \times n$ matrices $J = (J_b^a)$, $a, b, \ldots = 1, \ldots n$. These matrices parametrise a complex structure on the fibres of the bundle $j(R^{2n}) \to F$ over the point J, and define the antiholomorphic vector fields $\partial/\partial\bar{z}^a(J)$ on R^{2n} and $\bar{\partial}_J$-operator:

$$\bar{\partial}_J = d\bar{z}^a(J)\frac{\partial}{\partial\bar{z}^a(J)}, \quad \frac{\partial}{\partial\bar{z}^a(J)} = \frac{\partial}{\partial\bar{z}^a} + J_a^b\frac{\partial}{\partial z^b}, \tag{1}$$

where $z^a = x^a + iy^a$, (x^a, y^a) are coordinates in R^{2n}, and z^a are coordinates on C_0^n. Clearly, $\bar{\partial}_J^2 = 0$.

Consider the trivial Hermitian vector bundle E over the Euclidean space R^{2n}, associated with the principal G-bundle over R^{2n}, with connection which components are identified with the Yang — Mills (YM) potentials $A_1, \ldots, A_{2n}$. We shall denote by ψ the sections of the bundle $\tilde{E}$, which is the pull-back of the bundle E over R^{2n} to the manifold $j(R^{2n})$. They are functions $\psi(x, J)$ on $j(R^{2n})$ depending on $x \in R^{2n}$, $J \in F$ and taking values in the space of complex representation (e.g., C^N) of the algebra $\mathfrak{g}$.

Connection on a complex bundle E can be used to lift the operators $\bar{\partial}_J$ from R^{2n} to $j(R^{2n})$. We can introduce the structure of the holomorphic vector bundle in $\tilde{E}$ identifying the operator $\bar{\partial}$ on $\tilde{E}$ $(\bar{\partial}^2 = 0)$ with the (0,1)-component $\bar{D}$ of the connection on $j(R^{2n})$. In coordinate, a section ψ of the bundle $\tilde{E}$ is holomorphic if

$$(\bar{\partial}_a + J_a^b\partial_b + \bar{B}_a + J_a^b B_b)\,\psi(x, J) = 0, \tag{2}$$

$$\frac{\partial}{\partial\bar{J}_a^b}\psi(x, J) = 0, \tag{3}$$

where $B_1 = 2^{-1/2}(A_1 - iA_2), \ldots, B_n = 2^{-1/2}(A_{2n-1} - iA_{2n})$, $\bar{J}_a^b$ is a complex conjugation for J_a^b. Condition (3) is equivalent to the choice of complex coordinates on the manifold F and Eqs. (3) may be trivially satisfied for ψ depending on J_a^b and not depending on $\bar{J}_a^b$. The linear equations (2), defining the holomorphic structure in the bundle E, put some restrictions on the gauge fields B_a.

The compatibility condition of Eqs.(2) has a form:

$$F_{\bar{a}\bar{b}} + J_a^{\ c}F_{c\bar{b}} - J_b^{\ c}F_{c\bar{a}} + J_a^{\ c}J_b^{\ d}F_{cd} = 0, \tag{4}$$

where

$$F_{ab} = \partial_a B_b - \partial_b B_a + [B_a, B_b],\ F_{c\bar{b}} = \partial_c B_{\bar{b}} - \partial_{\bar{b}} B_c + [B_c, B_{\bar{b}}],$$

$$F_{\bar{a}\bar{b}} = (\overline{F_{ab}}),\ F_{\bar{c}b} = (\overline{F_{c\bar{b}}}).$$

By definition, Eqs.(4) are the generalized self-duality equations for the gauge fields in R^{2n}.

Now everything reduces to the choice of independent components $J_a^{\ b}$. By different choices of $J_a^{\ b}$ we shall obtain different linear systems, different self-duality equations and the embeddings of different integrable equations into the generalized self-duality equations (4).

Let us choose, for example, $n = 2k$ and $d = 2n = 4k$. Replace $a, b, \ldots$ by $(\mu i), (\nu j), \ldots$, where $\mu, \nu, \ldots = 1, 2; i, j, \ldots = 1, \ldots k$. Put

$$J_{(\mu i)}^{(\nu j)} = \lambda \varepsilon_\mu^\nu \delta_i^j, \tag{5}$$

where $\varepsilon_1^2 = -\varepsilon_2^1 = 1, \lambda \in CP^1$. Then Eqs.(2) are reduced to the equations

$$(\bar{\partial}_{x_i} + \lambda\partial_{y_i} + \bar{C}_i + \lambda D_i)\psi = 0,\ (\bar{\partial}_{y_i} - \lambda\partial_{x_i} + \bar{D}_i - \lambda C_i)\psi = 0, \tag{6}$$

where $\partial_{x_i} \equiv \partial_{1i}, \partial_{y_i} \equiv \partial_{2i}, C_i \equiv B_{1i}, D_i \equiv B_{2i}, i = 1, \ldots, k$. Let

$$\partial_{y_i}\psi = -\bar{\partial}_{x_{i+1}}\psi,\ \partial_{x_i}\psi = \bar{\partial}_{y_{i+1}}\psi, \tag{7}$$

and $\partial_{y_l}\psi = D_i = C_i = 0$ when $1 \le l < i \le k$. Then linear system (6) is reduced to the systems, considered in [2,6].

Now let

$$J_{(\mu i)}^{(\nu j)} = \delta_\mu^\nu J_i^j. \tag{8}$$

where $J_1^2 = -J_2^1 = \pi^2, J_1^3 = -J_3^1 = \pi^3, \ldots, J_1^k = -J_k^1 = \pi^k$, and other J_i^j equal zero. Then we have

$$
\begin{aligned}
&(\bar\partial_{\mu 1} + \pi^A \partial_{\mu A} + \bar B_{\mu 1} + \pi^A B_{\mu A})\psi = 0,\\
&(\bar\partial_{\mu 2} - \pi^2 \partial_{\mu 1} + \bar B_{\mu 2} - \pi^2 B_{\mu 1})\psi = 0,\\
&\dots\\
&(\bar\partial_{\mu k} - \pi^k \partial_{\mu 1} + \bar B_{\mu k} - \pi^k B_{\mu 1})\psi = 0,
\end{aligned}
\tag{9}
$$

where $A = 2, ..., k$. Let $\partial_{\mu 1}\psi = \bar\partial_{\mu 2}\psi = \dots = \bar\partial_{\mu k}\psi = 0$, $B_{\mu 1} = \bar B_{\mu 2} = \dots = \bar B_{\mu k} = 0$. Then Eqs.(9) are reduced to the equations, introduced by Ward [10]:

$$
\begin{aligned}
&(\partial_t + \pi^A \partial_{1A} + \bar B_{11} + \pi^A B_{1A})\psi = 0\\
&(\partial_x + \pi^A \partial_{2A} + \bar B_{21} + \pi^A B_{2A})\psi = 0
\end{aligned}
\tag{10}
$$

where $\partial_t \equiv \bar\partial_{11}$, $\partial_x \equiv \bar\partial_{21}$.

Finally, in (10) let $\pi^A = f^A(\lambda)$, where f^A are functions of $\lambda \in C$. It means that we consider one-dimensional complex submanifold B in the base F of the bundle $j(R^{2n}) \to F$ and the restriction $Z = j(R^{2n})|_B$ of this bundle on B. Then Eqs. (10) will define the holomorphic structure in the bundle $\tilde E$ over the twistor manifold Z.

We may embed the equations of any integrable model in (1 + 1) dimensions in Eqs.(10) if we put $\partial_{\mu A}\psi = 0$, choose the functions $f^A(\lambda)$, matrices $B_{\mu A}$, $\bar B_{\mu A}$ and a number k ($d = 4k$). For example, the Landau — Lifshitz equations may be obtained as a particular case of Eqs.(10) when $k = 7$.

If we take $\partial_y \equiv \partial_{1k}$, $\partial_{2A}\psi = 0$, $\partial_{1i}\psi = 0$ when $i \neq k$, $\partial_y\psi \neq 0$ and choose $\pi^A = \lambda^{A-1}$, then Eqs.(10) coincide with the equations of the integrable models in (2 + 1) dimensions, introduced in [6]. It is not clear now whether all the integrable equations in (2 + 1) dimensions may be embedded into Eqs.(10) or not. Apparently, this may be done if one will use the infinite dimensional Lie algebras (see, e.g., [5,21]). In any case, all integrable equations in (2 + 1) dimensions and many integrable equations in (2 + 1) dimensions can be obtained upon appropriate reduction of the generalized SDYM equations (4).

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